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# ASYMPTOTICS OF EXPANSION OF THE EVOLUTION OPERATOR KERNEL IN POWERS OF TIME INTERVAL $\Delta t$

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## Abstract

The upper bound for asymptotic behavior of the coefficients of expansion of the evolution operator kernel in powers of the time interval  $\Delta t$  was obtained. It is found that for the nonpolynomial potentials the coefficients may increase as  $n!$ . But increasing may be more slow if the contributions with opposite signs cancel each other. Particularly, it is not excluded that for number of the potentials the expansion is convergent. For the polynomial potentials  $\Delta t$ -expansion is certainly asymptotic one. The coefficients increase in this case as  $\Gamma(n \frac{L-2}{L+2})$ , where  $L$  is the order of the polynomial. It means that the point  $\Delta t = 0$  is singular point of the kernel.

# 1 Introduction

Since it was established, that the series of conventional perturbation theory in coupling constant for anharmonic oscillator and for number of models in quantum field theory are asymptotic ones [1]–[3], many attempts to construct some other representations, which would allow to take into account nonperturbative in coupling constant effects (see, e. g., [4]–[12]), were undertaken. Other direction of research is connected with attempts to obtain physically interesting information from analysis of divergent series by means of Borel summation method, Padé approximants etc. (summation of asymptotic series) [13]–[16]. Presence of a lot of different approaches means that the problem of nonperturbative calculations has not satisfactory solution till now.

In paper [17] attempt was made to calculate the evolution operator kernel in a form of the series in powers of time interval  $\Delta t$ . In this paper we examine the asymptotic behavior of that expansion.

Let us remind briefly the main relations, determining the expansion of the kernel in powers of  $\Delta t$  [17]. If the kernel  $\langle q', t' | q, t \rangle$  satisfies the Schrödinger equation

$$i \frac{\partial}{\partial t} \langle q', t' | q, t \rangle = -\frac{1}{2} \frac{\partial^2}{\partial q'^2} \langle q', t' | q, t \rangle + V(q') \langle q', t' | q, t \rangle \quad (1)$$

(here and everywhere below we use dimensionless variables, defined in standard manner, and for the sake of simplicity consider only one-dimensional case), then it may be written in a form

$$\langle q', t' | q, t \rangle = \left( \frac{-i}{2\pi\Delta t} \right)^{1/2} \exp \left\{ i \frac{(q' - q)^2}{2\Delta t} \right\} \left\{ 1 - \sum_{n=1}^{\infty} (i\Delta t)^n Y_n(q', q) \right\} \quad (2)$$

( $Y$ -representation), or

$$\langle q', t' | q, t \rangle = \left( \frac{-i}{2\pi\Delta t} \right)^{1/2} \exp \left\{ i \frac{(q' - q)^2}{2\Delta t} - \sum_{n=1}^{\infty} (i\Delta t)^n P_n(q', q) \right\} \quad (3)$$

( $P$ -representation). For the functions  $P_n$  and  $Y_n$  we have the recurrent relations

$$Y_1(q', q) + (q' - q) \frac{\partial Y_1(q', q)}{\partial q'} = V(q'), \quad (4)$$

$$nY_n(q', q) + (q' - q) \frac{\partial Y_n(q', q)}{\partial q'} = \frac{1}{2} \frac{\partial^2 Y_{n-1}(q', q)}{\partial q'^2} - V(q') Y_{n-1}(q', q), \quad (5)$$

$$\begin{aligned} nP_n(q', q) + (q' - q) \frac{\partial P_n(q', q)}{\partial q'} &= \\ &= \frac{1}{2} \frac{\partial^2 P_{n-1}(q', q)}{\partial q'^2} - \frac{1}{2} \sum_{n'=1}^{n-2} \frac{\partial P_{n'}(q', q)}{\partial q'} \frac{\partial P_{n-n'-1}(q', q)}{\partial q'}. \end{aligned} \quad (6)$$

And besides  $P_1(q', q) = Y_1(q', q)$ ,  $P_1(q', q') = V(q')$ . So, we can calculate the coefficient functions  $Y_n$  and  $P_n$  till any order  $n$  with a help of (4)–(6) proceeding from given potential  $V(q)$  and then substitute them into (2), (3). That will be formal solution of evolution equation (1).

The solutions of equations (4)–(6) may be represented in integral form

$$Y_n(q', q) = \int_0^1 \eta^{n-1} d\eta \Phi_n(q + (q' - q)\eta, q), \quad (7)$$

$$P_n(q', q) = \int_0^1 \eta^{n-1} d\eta F_n(q + (q' - q)\eta, q), \quad (8)$$

where

$$\Phi_n(x, q) = \frac{1}{2} \frac{\partial^2 Y_{n-1}(x, q)}{\partial x^2} - V(x) Y_{n-1}(x, q) \quad (9)$$

and

$$F_n(x, q) = \frac{1}{2} \frac{\partial^2 P_{n-1}(x, q)}{\partial x^2} - \frac{1}{2} \sum_{n'=1}^{n-2} \frac{\partial P_{n'}(x, q)}{\partial x} \frac{\partial P_{n-n'-1}(x, q)}{\partial x} \quad (10)$$

are the right hand sides of the equations (5) and (6), respectively. If  $n = 1$ , then  $F_1(x, q) = \Phi_1(x, q) = P_1(x, x) = V(x)$  (this is right hand side of (4)).

We will use representations (7)–(10) to analyze convergence of the series in (2), (3).

## 2 Study of asymptotics for $Y$ -representation

$Y$ -representation is more simple for analysis, because  $Y_n$  for given potential  $V$  depends on  $Y_{n-1}$  only, but not on all previous coefficient functions  $Y_i$  with  $i < n - 1$ , as it take place for  $P$ -representation. Let us consider in the beginning that the potential  $V(x)$  is nonpolynomial and all its derivatives are not equal to zero identically.

We can derive from (7), (9)

$$\begin{aligned} Y_n(q', q) &= \int_0^1 d\eta_1 \int_0^1 \eta_2 d\eta_2 \int_0^1 \eta_3^2 d\eta_3 \dots \int_0^1 \eta_n^{n-1} d\eta_n \times \\ &\times \left( -V(x_n) + \frac{1}{2} \frac{\partial^2}{\partial x_n^2} \right) \left( -V(x_{n-1}) + \frac{1}{2} \frac{\partial^2}{\partial x_{n-1}^2} \right) \dots \left( -V(x_3) + \frac{1}{2} \frac{\partial^2}{\partial x_3^2} \right) \times \\ &\times \left( -V(x_2) + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} \right) V(x_1). \end{aligned} \quad (11)$$

Here for brevity we denoted  $x_i = q + (x_{i+1} - q)\eta_i$ . Generally,  $x_i$  with different numbers  $i$  are connected as follows

$$x_i = q + (x_{i+l} - q)\eta_i \eta_{i+1} \dots \eta_{i+l-1}, \quad (12)$$

and derivatives, as it is seen from (12), are connected by the relations

$$\frac{\partial}{\partial x_i} = \eta_{i-l} \eta_{i-l+1} \dots \eta_{i-1} \frac{\partial}{\partial x_{i-l}}. \quad (13)$$

Disclosing brackets in (11) and reducing with a help of (13) all differentiations of the potential to differentiation of it with respect to full argument, i. e. to the form  $V^{(k)}(x_i) \equiv \frac{d^k}{dx_i^k} V(x_i)$ , we can get the expression for  $Y_n$  with following structure

$$\begin{aligned} Y_n(q', q) &= \int_0^1 d\eta_1 \int_0^1 \eta_2 d\eta_2 \int_0^1 \eta_3^2 d\eta_3 \dots \int_0^1 \eta_n^{n-1} d\eta_n \times \\ &\times \left\{ \sum_{k=1}^n \sum_{\{l_i\}} \kappa_{l_1, \dots, l_k}^{(k)} V^{(l_1)} \dots V^{(l_k)} \right\}. \end{aligned} \quad (14)$$

The sum  $\sum_{\{l_i\}}$  is taken over that sets of  $l_i$ , for which total order of derivatives is

$\sum_{i=1}^k l_i = 2(n-k)$ . Besides, one is to take into account the terms with different arguments of function  $V^{(l_i)}$ . The coefficients  $\kappa^{(k)}$  contain products of  $\eta_i$ .

Let us estimate  $Y_n$  using (14) and taking into account that, according to (11), the argument of the potential  $V$  belongs to the interval  $[q, q']$ . If we do not consider the singular potentials, then we can adopt that at finite interval every function  $V(x)$  with all its derivatives is bounded, i. e. for every number  $k$  the positive constant  $C_k$  exists, so that for all  $x$  from the interval  $[q, q']$  following condition is satisfied:  $|V^{(k)}(x)| < C_k$ . Then we have

$$\begin{aligned} |Y_n(q', q)| &\leq \\ &\leq \int_0^1 d\eta_1 \int_0^1 \eta_2 d\eta_2 \dots \int_0^1 \eta_n^{n-1} d\eta_n \left\{ \sum_{k=1}^n \left| \sum_{\{l_i\}} \kappa_{l_1, \dots, l_k}^{(k)} V^{(l_1)} \dots V^{(l_k)} \right| \right\} < \\ &< \int_0^1 d\eta_1 \int_0^1 \eta_2 d\eta_2 \dots \int_0^1 \eta_n^{n-1} d\eta_n \left\{ \sum_{k=1}^n C^k \left| \sum_{\{l_i\}} \kappa_{l_1, \dots, l_k}^{(k)} \right| \right\}. \end{aligned} \quad (15)$$

In (15) we denoted  $C \equiv \max_{\{k\}} C_k$ . It is hard to calculate  $\sum_{\{l_i\}} \kappa_{l_1, \dots, l_k}^{(k)}$  exactly, but it is possible to estimate it from above, putting in  $\kappa^{(k)}$  all  $\eta_i$  to be equal to 1. Then the coefficients  $N(n, k) = \sum_{\{l_i\}} \kappa_{l_1, \dots, l_k}^{(k)} (\eta_i = 1)$  can be calculated recurrently from the equation

$$N(n, k) = N(n, k-1) + \frac{1}{2} k^2 N(n-1, k), \quad (16)$$

which is direct consequence of (7), (9). Here  $N(n, k)$  is different from zero only if  $k$  varies from 1 to  $n$ , besides  $N(1, 1) = 1$ .

It is easy to see, that running recurrent equation (16) from up to down we get the main contribution for large  $n$  from the terms, corresponding to folloing path of transition from  $N(n, k)$  to  $N(1, 1)$ : at first we should fix  $k$  and decrease  $n$  by one unit per step till  $n$  becomes equal to  $k$ , then we should decrease both  $k$  and  $n$  by one unit per step till  $n$  and  $k$  become equal to 1. This leads to estimate

$$N(n, k) \sim \left( \frac{k^2}{2} \right)^{n-k}. \quad (17)$$

Let us find the number  $k$ , corresponding to the terms, giving the main contribution to (14) at large  $n$ . For this we test the right hand side of (17) for an extremum as function of  $k$  (temporary we consider  $k$  real) at fixed sufficiently large  $n$ . The value  $k_m = [k]$  ( $[\dots]$  is integer part of number), where  $k$  is the root of equation

$$k(2 \log k + 2 - \log 2) = 2n, \quad (18)$$

corresponds to maximum contribution. Equation (18) has one root, which may be calculated approximately with a help of relation

$$k \approx \frac{n}{\log n} \left( 1 + \frac{\log \log n}{\log n} \right). \quad (19)$$

So, the main contribution is determined by the coefficient

$$N(n, k_m) \sim \frac{n^{2n}}{(2e^2)^n (\log n)^{2n}}. \quad (20)$$

We can now estimate  $Y_n$  proceeding from (20) and taking into consideration that integration over  $\eta_i$  in (15) gives the factor  $1/n!$ :

$$|Y_n(q', q)| \lesssim \frac{1}{n!} C^{k_m} N(n, k_m) \sim \frac{n!}{n} \frac{C^{\frac{n}{\log n}}}{2^n (\log n)^{2n}} \sim \frac{n!}{n 2^n (\log n)^{2n}}. \quad (21)$$

This evaluation is overestimated, because we put  $\eta_i = 1$  in  $\kappa^{(k)}$  and because we do not consider in (14) possible cancellations of the terms, corresponding to the same number of multipliers  $k$ . Cancellations may occur thanks to different signs of derivatives  $V^{(l_i)}(x)$  for different orders  $l_i$ . Uncertainty of the estimate of the first kind can be evaluated from consideration in (14) the terms of form

$$\frac{\partial^{2(n-1)} V(x_1)}{\partial x_n^2 \partial x_{n-1}^2 \dots \partial x_2^2} = \eta_{n-1}^2 \eta_{n-2}^4 \dots \eta_1^{2n} V^{(2(n-1))}(x_1),$$

which give the minimal contribution when dependence of  $\kappa^{(k)}$  on  $\eta_i$  is taken into account. Instead of factor  $1/n!$  in (21) that term gives

$$\frac{n!}{(2n)!} \sim \frac{\sqrt{n}}{n! 4^n}.$$

So, correct estimate lies at the bounds

$$|Y_n(q', q)| \lesssim \left( \frac{\sqrt{n}}{4^n} \div 1 \right) \frac{n!}{n} \frac{C^{\frac{n}{\log n}}}{2^n (\log n)^{2n}}. \quad (22)$$

This correction does not touch the main factor  $\frac{n!}{(\log n)^{2n}}$ , which means, that if contributions of different signs does not cancel each other essentially, then the representation (2) is asymptotic one. Nevertheless, we do not state, that for every potential expansion will be divergent. On the contrary, one may hope, that that functions  $V(x)$  exist (besides squared potentials), for which in terms in (14) essential cancellations take place and the coefficient functions  $Y_n$  rise more slowly, then  $n!$ .

Now we consider the polynomial potentials. It was supposed in the estimates made above, that derivatives of the potentials of all orders do not equal to zero identically. This is not so for polynomial potentials. So, it is necessary to modify our reasonings.

Let  $V(x)$  is polynom of order  $L$ . Then  $V^{(L+1)}(x) \equiv 0$ . We will consider the terms in (14) containing  $k$  multipliers  $V^{(l_i)}$ . Because  $\sum_{i=1}^k l_i = 2(n-k)$  and the order of derivative, acting on every multiplier, is not higher then  $L$ , then this term is not equal to zero only if  $2(n-k) \leq kL$  or  $k \geq \frac{2n}{L+2}$ . So, there exists the down bound for possible values of  $k$ , which is higher, at large  $n$ , then  $k_m$  obtained from (19). In case of the polynomial potential one has to take  $k_m$  in a form

$$k_m = \left\lceil \frac{2n}{L+2} \right\rceil. \quad (23)$$

Then we find from (17) taking into account (23)

$$N(n, k_m) \sim \sqrt{n} \left( \frac{e^2}{2L^2} \right)^{\frac{nL}{L+2}} \Gamma \left( n \frac{2L}{L+2} \right). \quad (24)$$

And for  $Y_n$  we have estimate, founded on (24):

$$|Y_n(q', q)| \lesssim \left( \frac{(2e^2)^L C^2}{(L-2)^{(L-2)} (L+2)^{(L+2)}} \right)^{\frac{n}{L+2}} \Gamma \left( n \frac{L-2}{L+2} \right). \quad (25)$$

It turned out to be that for the polynomial potentials the representation (2) is asymptotic one. But the coefficient functions rise with growth of  $n$  more slowly, then  $n!$ . Particularly, for  $L = 4$  we have from (25)  $|Y_n| \sim \Gamma(n/3)$ . For harmonic oscillator, as it is naturally,  $|Y_n| \sim \left(\frac{e}{4}\sqrt{2C}\right)^n$ , i. e. the series in (2) has finite convergence range.

### 3 Asymptotics for $P$ -representation

Let us consider now the representation (3). Analogously to previous case we introduce the coefficients  $N(n, k)$ , which satisfy the relation

$$N(n, k) = \frac{1}{2}k^2N(n-1, k) + \frac{1}{2} \sum_{n'=1}^{n-2} \sum_{k'=1}^{k-1} k'(k-k')N(n', k')N(n-n'-1, k-k'), \quad (26)$$

following from (18). Here  $N(1, 1) = 1$  and  $N(n, k)$  is different from zero only if  $1 \leq k \leq \left[\frac{n+1}{2}\right]$ . The main contribution at large  $n$  comes from the terms, which arise in running of (26) from up to down when the numbers  $n, k$  are changed in following manner: at the beginning we fix  $k$  and decrease  $n$  till  $n$  becomes equal to  $2k$ , then we decrease  $n$  by two units and  $k$  by one unit per step till  $n$  and  $k$  become equal to 1. It gives:

$$N(n, k) \sim \frac{k^{3/2}}{2^{n-k}} k^{2n-3k}. \quad (27)$$

Maximal contribution will be got from the terms, containing  $k_m = [k]$  multipliers  $V^{(l_i)}$ , where  $k$  is determined by equation

$$k(3 \log k + 3 - \log 2) = 2n. \quad (28)$$

Equation (28) has only one root, which is approximately (for large  $n$ ) equal to

$$k \approx \frac{2}{3} \frac{n}{\log n} \left(1 + \frac{\log \log n}{\log n}\right). \quad (29)$$



Then asymptotically main contribution is determined, as is seen from (27), (29), by the coefficient

$$N(n, k_m) \sim \frac{n^{2n+3/2}}{(2e^2)^n (\log n)^{2n}}. \quad (30)$$

When we considered  $Y$ -representation, calculation of the integrals of kind  $\int_0^1 \eta_i^{i-1} d\eta_i$  led to the factor  $1/n!$  (see (15), (21)). In this case situation is rather complicated. When we express by means of (6) the coefficient  $P_n(q', q)$  through lower coefficient functions, decreasing  $n$  as it was described above, then the integral

$$\begin{aligned} & \int_0^1 \eta_n^{n-1} d\eta_n \int_0^1 \eta_{n-1}^{n-2} d\eta_{n-1} \dots \int_0^1 \eta_{2k_m}^{2k_m-1} d\eta_{2k_m} \times \\ & \times \int_0^1 \eta_{2k_m-1}^{2k_m-2} d\eta_{2k_m-1} \int_0^1 \eta_{2k_m-3}^{2k_m-4} d\eta_{2k_m-3} \dots \int_0^1 \eta_3^2 d\eta_3 \int_0^1 \eta_1^0 d\eta_1 \end{aligned}$$

arises. It gives for estimate of  $P_n$  the factor

$$\frac{2^{k_m-1} (k_m - 1)!}{n!}. \quad (31)$$

Involving (31) we get following evaluation

$$|P_n(q', q)| \lesssim \frac{2^{k_m} k_m!}{n!} C^{k_m} N(n, k_m) \sim \frac{n^{1/2} n!}{(\log n)^{2n}} \left( \frac{e^{2/3}}{2} \right)^n. \quad (32)$$

This evaluation, so as (21), is overestimated. Asymptotic increase of type  $n!$  will take place only if there is no essential cancellations of the contributions with different signs.

Consider the potentials which are the polynomials of order  $L$ . In this case almost all previous reasonings are right, excluding one point. For  $k_m$  one should take expression (23) instead of (29). Then we have evaluation

$$|P_n(q', q)| \lesssim n^2 2^n \left( \frac{e^{2(L-1)} C^2}{(L-2)^{(L-2)} (L+2)^{(L+2)}} \right)^{\frac{n}{L+2}} \Gamma \left( n \frac{L-2}{L+2} \right). \quad (33)$$

The main asymptotic behavior in (33), so as in (25), is determined by the factor  $\Gamma\left(n\frac{L-2}{L+2}\right)$ . So, for  $L > 2$  the series in  $P$ -representation are asymptotic one too.

Note, that in estimates (25), (33) for the polynomial potentials, maybe, the factors of type  $n^a b^n$ , are some overestimated, but the main rise of type  $\Gamma\left(n\frac{L-2}{L+2}\right)$  is established exactly. For the nonpolynomial potentials estimates (21), (22), (32) determine only upper bounds for possible asymptotic rise. They can be not achieved for some potentials.

## 4 Conclusion

As it is seen from (25), (33), in the case of polynomial potentials the coefficient functions in expansions (2), (3) rise as  $n^a b^n \Gamma\left(n\frac{L-2}{L+2}\right)$  for  $n \rightarrow \infty$ . So, for  $L > 2$  the series in (2), (3) are asymptotic ones. We can note here, that divergence of expansion in  $\Delta t$  means that the point  $\Delta t = 0$  is singular point of the kernel. Discussion of this topic can be found in [18].

If the potential is nonpolynomial continuous function, then for asymptotics of the coefficient functions one has (see (22), (32)) an upper bound of kind  $n^a b^n \frac{n!}{(\log n)^{2n}}$ . Nevertheless, this bound was obtained by summation of all contributions without taking into account the signs of the terms. So, real asymptotic rise for some potentials may be essentially more slow. Particularly, possibility of existing of number potentials, for which the expansion in  $\Delta t$  is convergent, is not excluded.

Note, that all presented reasonings may be easily generalized on many-dimensional case, because the representations of type (2)–(10) for  $D$ -dimensional space can be derived from ones for one-dimensional space by changing of one-dimensional space variables to vector variables and by changing of derivatives over coordinates to  $D$ -dimensional operator *grad*. It leads to obvious changes in equations (16), (26) for estimates of asymptotics: instead of factor  $k^2$  one should take  $Dk^2$ , and instead of  $k'(k - k')$  one should take  $D^2 k'(k - k')$ . That substitution modifies final results not essentially: in (20), (21), (22), (24), (25), (30), (32), (33) one is to add the factor  $D^n$ . It does not affect the qualitative results obtained above.

In conclusion, let us pay attention to one feature of the representations (2)–(10). When we calculate the transition matrix element  $\langle \vec{q}', t' | \vec{q}, t \rangle$ ,

only values of the potential  $V(\vec{x})$  in the points, lying on the segment, joining the points  $\vec{q}$  and  $\vec{q}'$ , are essential. Behavior of the potential in other points does not affect on value of the kernel at all!

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